Uniwersytet
Kardynała Stefana Wyszyńskiego
w Warszawie

Towards a Contemporary Ontology

The New Dual Paradigm in Natural Sciences: Part I

Module 2 Class 3: The issue of the foundations of mathematics

Course WI-FI-BASTI1

2014/15

Introduction

Class 3: “The issue of the foundations of mathematics from Riemann to Gödel and the Hilbert second problem”

Course modules

Modules	Topic	Suggested Readings
	SECTION ONE	
0.	*Introduction and Course Overview*	
1.	The Birth of Modern Science	Refs.: **1**, chs. 0, 1, 2.
2.	The Question of Truth in Modern Science	Refs.: **1,** chs. 3, 4.
	SECTION TWO	
3.	The Information Theoretic Interpretation of QM	Refs.: **4-10**.
4.	QFT interpretation as ‘Second Quantization’ and the Physics of the Condensed Matter	Refs.: **11**, pp. vi-xii, 1-35, 137-178; **12; 13**.
	SECTION THREE	
5.	The (Co)Algebraic Interpretation of QFT as q-deformed Hopf Algebra / Coalgebra	Refs.: **11,** pp. 131-185; **14**.
6.	The DDF Principle of QFT, its Cosmological Relevance and Its Ontological Interpretation	Refs.: **14-19**; **1**, ch. 5.
	SECTION FOUR	
7.	Universal Coalgebra and the Interpretation of QFT Systems as STS	Refs.: **16**; **20**
8.	*Conclusions*	

Main Contents of the Module 2

- This module 2 is mainly concerned with the problem of **truth** and/or the **problem of foundation** in pure and applied modern mathematical sciences**.** A problem emerging from the discovery of the **hypotetical nature of mathematics,** from which the, necessity derives of proving the consistency of mathematics – geometries and arithmetics, before all – given that the validity of the demonstrations is no longer granted by the soundness of the axioms. A particular subset of this more general problem concerns the **soundness** of the mathematical laws of physics.
- The four classes of this module concern**:**
  1. **The Greek origins of the Western science**.
  2. **The issue of the V postulate of Euclidean geometry in the history of Western mathematics**
  3. **The issue of the foundations of mathematics from Riemann to Gödel and the Hilbert second problem**
  4. **The issue of the Skolem paradox and the relativity principle in the algebraic interpretation of set theory**

Bibliography

Bibliography of the Class 3 of the Module 2

Bibliography

- Main References for this Module
  1. G. Basti, *Philosophy of Nature and of Science. Volume I: The Foundations,* transl. by Philip Larrey, Rome 2012 (for student use only), ch. 0,1,2, 3 [attached].
  2. C. B. Boyer, *A history of mathematics,* J. Wiley & Sons, New York, 1968 (available online at: https://archive.org/details/AHistoryOfMathematics (*Second Edition, ed. by U. C. Merzbach,* J. Wiley & Sons, New York, 1991).
- Other Reference:
  - E. Nagel, J. R. Newmann, *Gödel's proof. Revised edition,* Ed. by D. R. Hofstadter, New York UP, New York, 2001 (1958 First edition available online at the Internet Archive of the University of Florida: https://archive.org/details/gdelsproof00nage).
  - M. Hallett, *Cantorian set theory and limitation of size,* Clarendon Press, Oxford, 1984.

Class 3

The issue of the foundations of mathematics from Riemann to Gödel and the Hilbert second problem

The problem of the existence of infinite mathematical objects

- The abandon of the naive supposition of the absolute truth of mathematical objects typical of the early modern interpretation of calculus → problem of the **proof** of the existence **(soundness)** of the mathematical objects, even the **infinite ones of calculus,** through a proper **meta-mathematical theory.**

- The core notion of a metalanguage – even though the distinction language/metalanguage in mathematics is due to the successive work of Hilbert – is identified in the notion of **set *(Menge)***, starting from the essential works on the **calculus foundations**, and on the **convergent series of real valued functions** of **Karl Weierstrass** (1815-1897) (Weierstrass theorem, Bolzano-Weierstrass theorem, Stone-Weierstrass theorem, etc.).

- The core problem of the calculus foundations as to the **existence of infinite objects** (infinite sets, or sets with the cardinality of the universal class **V**) in mathematics is perfectly synthesized in this quotation of Georg Cantor (1845-1918):

The problem of actual infinity in mathematics

- Quotation starts: «Undoubtedly, we cannot avoid the existence of *variable* quantities in the sense of **potential infinity**; from this, one can demonstrate the necessity of **actual infinity**. For a variable quantity to exist in a mathematical theory, the «domain» of its variability must be known in advance, through a definition. However, this domain must not be, in turn, something variable, otherwise we lose any sound basis for the study of mathematics. Therefore, this domain is **a defined, actually infinite set of values**» Quotation ends (Cantor 1886, 9).

- Cantor attempt of a **constructive «piecewise» foundation of infinite sets,** even the **continuum** itself, through its powerful **Cantor theorem → discovery of the antinomies** as to the constructive demonstration of existence of the **universal set** (in the realm of cardinal numbers), and/or of the **maximal set** (in the realm of ordinal numbers) by the Burali-Forti antinomy → abandon of **Cantor's set theory** for different **axiomatic set theories (ZF, ZFC, NGB, NF, NFU…),** all characterized by the introduction of **axioms** for limiting the size of costructive sets, so to transform the **antinomies** into as many **paradoxes** (Hallett, 1984).

Russell's antinomy and the logical character of antinomies

- **Logicist** interpretation of foundations of mathematics through G. Frege (1848-1925) attempt of avoiding the antinomies by the foundation of **number** not on the notion of **set,** but on the notion of **class** (**class of classes),** by using his **generalized abstraction principle** («each property (predicate) determines the domain of the elements that satisfy it (it determines predicate 'extension'))» that in its axiomatic version (**axiom of comprehension**) by Russell reads:

$$\forall x \exists y \left(x \in y \equiv \varphi x \right)$$

- The discovery of **Russell's antinomy** reveals the **logical character of antinomies**, as well its **syntactic** and not **semantic** nature (like in the classical version of logical antinomies, e.g., the famous «liar's antinomy» well known since the disputes of Plato and Aristotle with the Sophist philosophers), as far as based on the notions of **membership** and of **inclusion** common also to the notion of **set** → solution of the antinomy by B. Russell's (1872-1970) **theory of types** in the *Principia,* and later by W.V.O. Quine (1908-2000) **New Foundations (NF) theory** with the **stratified character** of membership predication.
- The syntactic nature of antinomies was definitively made clear by Gödel **incompleteness theorems** and of their extensions in **computability theory** as unavoidable **limitation theorems** of any formalized language, also because they definitively **shelved Hilbert's formalistic program** on foundations.

Hilbert’s formalist program for the problem of foundations

- Just as the main contribution of G. Frege was not its logistic program for the problem of foundations, but the development of the formalism of the **mathematical logic,** based on the extension of the mathematical notion of function to logic by the notion of **propostional function,** formally **unifying the classical sentential and predicate logic**, and making of him the most influential logician in the history of Western thought, after Aristotle and Leibniz, and before Gödel,

- So, the main contribution of the **formalist program** of D. Hilbert (1862-1943) was not for the solution of the foundatons problem, but for its rigorous definition of **modern axiomatic method** in the scientfic enquiry.

- Hilbert’s choice consisted thus in separating the logical-formal procedures of **demonstration and inference** (= syntax, logical calculus as symbol manipulation by rules) from the **proof of their consistency and truth** (= semantic, meta-logic justification and/or interpretation of a calculus) → implicit recovering on a rigorous formal basis:

  1. The Scholastic theory of ***suppositio*** (reference) of given terms/predicates, with the distinction between:
     - ***Suppositio materialis*** or **grammatical self-reference** of any well-formed, i.e., unambiguous, logical symbol;
     - ***Suppositio formalis*** or **reference** of logical symbols and/or predicative sentences, either **intra-linguistic** (logic) or **extra-linguistic** (ontology);
  2. The Scholastic distinction between ***logica minor* (calculus)** and ***logica maior* (symbol constitution)** without the ambiguities between logic and epistemology that such distinctions had in the modern transcendental philosophy of logic.

Hilbert's axiomatization of geometry and its systematic use of a recursive method of proof

- After his early study on algebraic invariants, in which he had demonstrated the potential of a **recursive (finite) method of proof**, one of the first and more fundamental results of Hilbert's work was the famous demonstration of the calculus of distances, which constituted the core of Hilbert's treatise on the **Foundations of Geometry (1898-99).** This work aimed at demonstrating the independence of pure geometry from numbers, by showing that all (Euclidean) plane geometry can be represented as a particular, abstract algebraic structure: the **commutative field**.
- Therefore, the core of Hilbert's results on the foundations of the Euclidean geometry – in this similar to Riemann's result for non-Euclidea geometry – was the demonstration that analytical geometry **does not need to presume the existence of numbers in order to justify the use of algebra**.

The core of Hilbert’s formalistic interpretation as to Frege’s logicistic one

- In an epistolary exchange with Frege (Webb 1980, 87) it is summarized the deep difference between the two, Frege’s and Hilbert’s foundational approaches:
  - **Frege**: «Are there ways, perhaps, to demonstrate the consistency other than to exhibit an object that possesses all the properties? Anyway, if one has such an object at their disposal, one has no need to demonstrate its existence through the circular demonstration of its consistency » (i.e., existence and truth → consistency).
  - **Hilbert:** «This procedure of construction of an axiom by referring to its truth, and concluding from this that it is compatible with other defined concepts, is precisely the primary source of mistakes and confusion. If the arbitrarily established axioms do not contradict one another, then they are true, and the objects defined through them exist. For me, this is the one criterion of truth and existence» (i.e., consistency → truth and existence).
- → Hilbert’s demonstration of the **isomorphism** between Euclidean **geometric** structures/ /relations, and **algebraic** structures/relations (fields) defined on **real numbers** was aimed:
  - Not at the **semantic** goal of Riemann’s demostration of the **relative consistency of** non-Euclidean geometries by their Euclidean modeling, based on the **intuitive truth** of the Euclidean statements,
  - But at the **syntactic** task of demonstrating the **relative consistency** of geometries as to the consistency of **algebraic structures of arithmetics** as defined on sets of real numbers**.**

From Hilbert's formalistic program to Gödel incompleteness theorem

- Hilbert's result → centrality of the **foundations of arithmetics** for proving the consistency of modern mathematics despite the hypothetical character of its demonstration → **2° Hilbert problem** (of his famous list of 23 problems): «prove that the axioms of arithmetics are consistent», where the axioms of arithemtics are the **Peano axioms** in the *Principia Mathematica* (PM) formal language.
- The solution consists in the possibility of proving the **absolute consistency of arithmetics** by a **finite procedure,** i.e. by proving the consistency of **formalized arithmetics** from within arithmetics, in a similar way to which in *Principia Mathematica* (PM) it was possible to give an **absolute consistency proof** of the **calculus of propositions** from within it**.**
  - The main difference is that, while the consistency of sentential calculus does not require any proof of its **completeness,** such a proof is a necessary condition in the case of arithmetics because it is a (first order) **theory on numbers** and not only a calculus → necessity of proving its **completeness:** all the true propositions in it **must be proved** as formally derived from its axioms.

Gödel incompleteness theorems and the two intermediate steps toward this historical result

1. The first step of such a formalistic Hilbert program was performed by K. Gödel by his demonstration of the **completeness of first order predicate calculus** (1929).

2. The second step is genious **Gödel arithmetization of any formal language (Gödel coding)** by using the theory of primes for assigning a **univocal numerical value** to whichever formula of a formalized language **(= Gödel numbering).**

- → Two **Gödel incompleteness theorems**:
  1. Any effectively generated theory capable of expressing elementary arithmetics in its Peano's formalization (PA), **cannot be both consistent and complete**. In particular, for any consistent, effectively generated formal theory that proves certain basic arithmetic truths, there is an arithmetical statement that is true, but not provable in the theory (Kleene 1967, p. 250)
  2. For any formal effectively generated theory *T* including basic arithmetical truths and also certain truths about formal provability, if *T* includes a statement of its own consistency, then *T* is inconsistent.

- Because of the universal applicability of **Gödel coding** to whichever formalized language/theory → **Gödel theorems** can be interpreted as **universal limitation theorems** for any formalized language/theory (i.e., any interpretation of the first order predicate calculus) → fundamental limitation theorems of Tarski, Skolem and Turing.