Uniwersytet
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w Warszawie

Towards a Contemporary Ontology

The New Dual Paradigm in Natural Sciences: Part I

Module 2 Class 3: The issue of the foundations of mathematics

Course WI-FI-BASTI1 2014/15

Introduction

Class 3: “The issue of the foundations of mathematics from Riemann to Gödel and the Hilbert second problem”

Course modules

Modules	Topic	Suggested Readings
	SECTION ONE	
0.	*Introduction and Course Overview*	
1.	The Birth of Modern Science	Refs.: **1**, chs. 0, 1, 2.
2.	The Question of Truth in Modern Science	Refs.: **1,** chs. 3, 4.
	SECTION TWO	
3.	The Information Theoretic Interpretation of QM	Refs.: **4-10**.
4.	QFT interpretation as 'Second Quantization' and the Physics of the Condensed Matter	Refs.: **11**, pp. vi-xii, 1-35, 137-178; **12; 13**.
	SECTION THREE	
5.	The (Co)Algebraic Interpretation of QFT as q-deformed Hopf Algebra / Coalgebra	Refs.: **11,** pp. 131-185; **14**.
6.	The DDF Principle of QFT, its Cosmological Relevance and Its Ontological Interpretation	Refs.: **14-19**; **1**, ch. 5.
	SECTION FOUR	
7.	Universal Coalgebra and the Interpretation of QFT Systems as STS	Refs.: **16**; **20**
8.	*Conclusions*	

Main Contents of the Module 2

- This module 2 is mainly concerned with the problem of **truth** and/or the **problem of foundation** in pure and applied modern mathematical sciences**.** A problem emerging from the discovery of the **hypotetical nature of mathematics,** from which the, necessity derives of proving the consistency of mathematics – geometries and arithmetics, before all – given that the validity of the demonstrations is no longer granted by the soundness of the axioms. A particular subset of this more general problem concerns the **soundness** of the mathematical laws of physics.
- The four classes of this module concern**:**

1. **The Greek origins of the Western science**.
2. **The issue of the V postulate of Euclidean geometry in the history of Western mathematics**
3. **The issue of the foundations of mathematics from Riemann to Gödel and the Hilbert second problem**
4. **The issue of the Skolem paradox and the relativity principle in the algebraic interpretation of set theory**

Bibliography

Bibliography of the Class 3 of the Module 2

Bibliography

- Main References for this Module
  1. G. Basti, *Philosophy of Nature and of Science. Volume I: The Foundations,* transl. by Philip Larrey, Rome 2012 (for student use only), ch. 0,1,2, 3 [attached].
  2. C. B. Boyer, *A history of mathematics,* J. Wiley & Sons, New York, 1968 (available online at: https://archive.org/details/AHistoryOfMathematics (*Second Edition, ed. by U. C. Merzbach,* J. Wiley & Sons, New York, 1991).
- Other Reference:
  - E. Nagel, J. R. Newmann, *Gödel's proof. Revised edition,* Ed. by D. R. Hofstadter, New York UP, New York, 2001 (1958 First edition available online at the Internet Archive of the University of Florida: https://archive.org/details/gdelsproof00nage).
  - M. Hallett, *Cantorian set theory and limitation of size,* Clarendon Press, Oxford, 1984.

Class 3

The issue of the foundations of mathematics from Riemann to Gödel and the Hilbert second problem

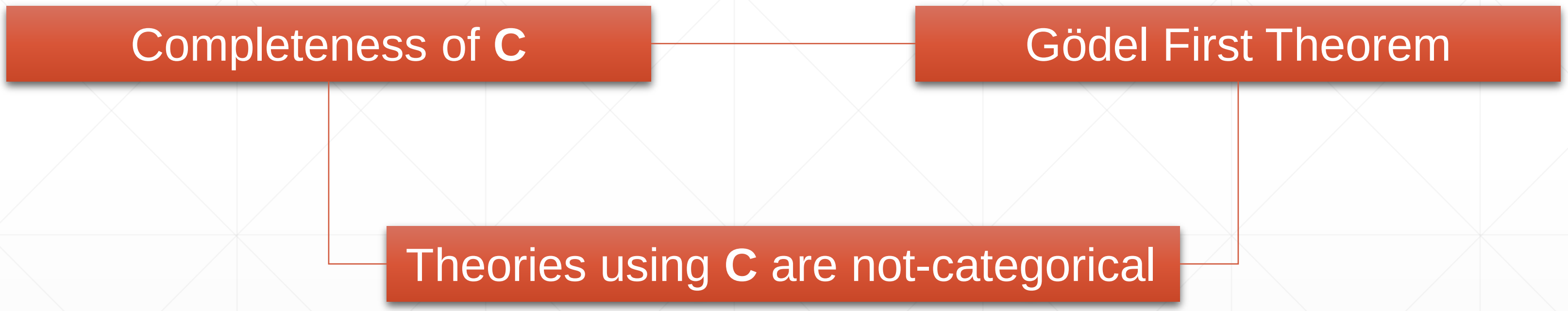
Axiomatic set theory as theory of foundations in the light of Gödel, Skolem, and Dedekind theorems

- There exists a sort of **indetermination relationship** between the demonstrative power and the expressive power of formalized theory depending on a corollary of Gödel incompleteness theorems, even though demonstrated before them (1915-1923), that is the **Löwenheim-Skolem theorem**:
  - «If **a first-order countable (i.e., consistent with the first-order predicate calculus, C)** theory has an infinite model, it is **a countable infinite model**» → any consistent first order theory is **non-categorical,** that is, it has no unique infinitary model **up to isomorphism,** i.e., all its models are **not isomorphic** among them and hence do not belong to the same (**logical) category**.
  - → **Löwenheim-Skolem paradox:** the Cantor set theory and **Cantor theorem** → distinction between **countable infinite sets** (bijective with the set of natural numbers) and **non-countable infinite sets** (the set of real numbers, the set of complex numbers, the set of all subsets of the natural numbers, etc.) cannot be expressed in **any first-order axiomatic set theory,** despite all the axioms of ordinary set theories are **first-oder sentences**.
  - On the other hand, because of **Dedekind theorem** («any infinite set is similar to (can be put in biunivocal correspondence with) a proper part of itself»), it is possible to demonstrate **the completeness of second-order arithmetics.**
  - → Axiomatic set theory as **theory of foundations of mathematics** (like whichever foundation theory) must be formalized using **higher order logic** with its full semantics and hence with **non-countable models** (Zermelo for ZFC, and Von Neumann for NGB both admit the existence of non-countable models only in the second order interpretations of ZFC and NGB set theories, respectively) → **finitary character** of any first-order theory.

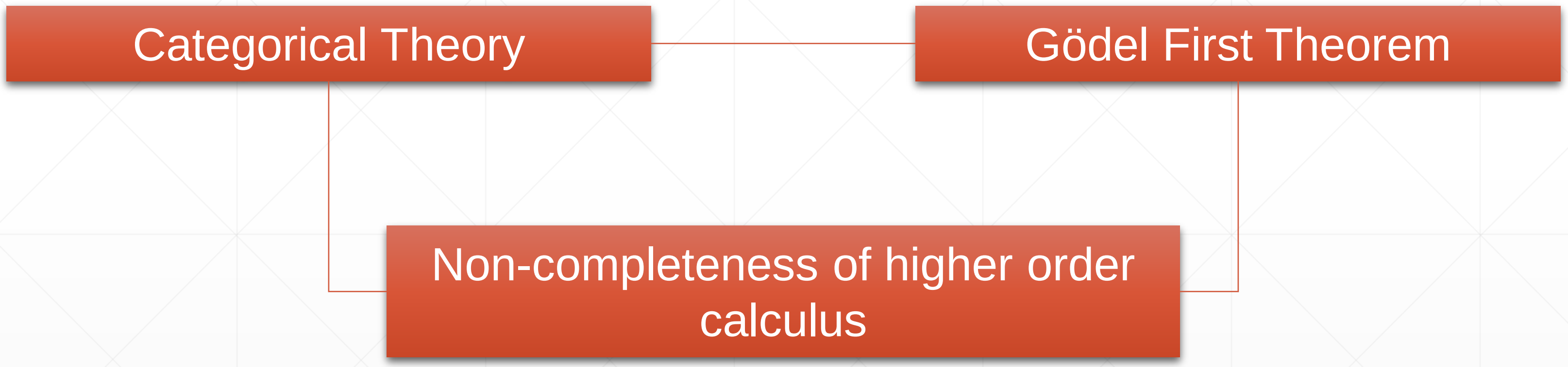
The trade-off between demonstrative power and categorical power of a formalized theory in the light of Gödel, Skolem, and Dedekind theorems

- → **Indetermination relationship consistency/categoricity (De Giorgi):**
  - If a formalized theory is based on **a powerful demonstrative method** (first order predicate calculus, **C**), it loses **explicative unity,** i.e., it is **not categorical** (e.g., first order axiomatic set theory).
  - If a formalized theory is **categorical,** i.e., endowed with a strong **unitary explicative power,** it is based on **a weaker demonstrative method** (higher order predicate calculus).
- **To sum up:**

Completeness of the calculus as to the categorical character of a formalized theory I



Completeness of the calculus as to the categorical character of a formalized theory II



The Skolem paradox and the relativity principle in the algebraic interpretation of set theory

- In (Skolem 1922) Skolem himself gave an interpretation of the **relevance** of his theorem for the axiomatic set theory according to his **algebraic or model-theoretic interpretation** of it:

1. The axiomatic set theory **cannot be a suitable theory of foundations of mathematics** since essential notions such as non-countable sets cannot be formalized in it.
   - Effectively this interpretation is not true because it is sufficient to use a **second-order semantics** in axiomatic set theory for granting the existence of **non-countable models** to axiomatic set theories such as ZFC and NGB → **Skolem paradox**, from the mathematical logic standpoint is only interesting as revealing «a novel and unexpected feature of formal systems» (van Heijenoort 1967) with no unavoidable antinomy for set-theoretical foundations of mathematics.
2. The notions of **countable/non-countable sets** in any algebraic or first-order model-theoretic interpretation of set theory is **relative to the algebra of sets** we are using. This interpretation is **true in two senses:**
   - What the Skolem theorem forbids to set theoretic semantics is an **absolute representations of non-countable objects** such as real numbers, in the sense that the set of **all real numbers** is not countable, but this does not imply that **subsets of them** can be posed in a bijective relation with natural numbers (Bays 2014), as effectively the same **diagonal method,** used by Cantor for demonstrating that the full set of them is non-countable, demonstrates.
   - In **category theory** it is possible to develop for algebras, **coalgebraic model-theoretic semantics** giving Skolem relativity principle an unexpected **constructive relevance,** both in mathematical and philosophical logic, as we see.