Uniwersytet
Kardynała Stefana Wyszyńskiego
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Towards a Contemporary Ontology

The New Dual Paradigm in Natural Sciences: Part II

Module 6: The formal ontology of the natural realism (NR) II

Course WI-FI-BASTI2

2014/15

Introduction

Module 6: ”The formal ontology of the natural realism (NR) II: Aczel sets and coalgebraic modal logic”

Course modules

Modules	Topic	Suggested Readings
	SECTION ONE	
0.	*Introduction and Course Overview*	
1.	QFT: an evolutionary interpretation of nature from cosmology to neuroscience	Refs.: **1-5.**
2.	QFT in fundamental physics and the Aristotelian-Thomistic ontology of nature	Refs.: **6**, chs. 5-6; **7-8.**
	SECTION TWO	
3.	Formal philosophy and formal ontology	Refs.: **9-11.**
4.	The formal ontology of the conceptual natural realism (CNR)	Refs.: **12-15.**
	SECTION THREE	
5.	The formal ontology of the natural realism (NR) I: logical vs. causal inference	Refs.: **16-18.**
6.	The formal ontology of the natural realism (NR) II: Aczel sets and coalgebraic modal logic	Refs.: **19-24.**
	SECTION FOUR	
7.	The formal ontology of the natural realism (NR) III: the duality logical/ontological truth	Refs.: **24-28.**
8.	*Conclusions*	

Bibliography

Bibliography of the Module 6

Bibliography

- **Main References**:

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2. L. S. Moss, "Non-wellfounded set theories", in: *Stanford Encyclopedia of Philosophy,* http://plato.stanford.edu/archives/win2014/entries/nonwellfounded-set-theory/ [accessed 20/1/2015]

- **Other references:**

3. P. ACZEL, «Non-well-founded sets», *CLSI Lecture Notes, vol.14*, 1988 [attached].

4. P. ACZEL & N. P. MENDLER, «A Final Coalgebra Theorem», in *Category Theory and Computer Science*, Lecture Notes in Computer Science , Springer , London, UK, 1989, CCCLXXXIX, 357-65.

Bibliography II

5. J. ADAMEK, “Introduction to coalgebras”, *Theory and Applications of Categories,* Vol. 14, No. 8, 2005, pp. 157–199. [attached]

6. JEAN-PIERRE MARQUIS, «Category Theory», in *Stanford Encyclopedia of Philosophy* <http://plato.stanford.edu/archives/sum2013/entries/category-theory/> [accessed 13 September 2014]

7. Y. VENEMA, «Algebras and co-algebras», in *Handbook of modal logic*, P. BLACKBURN, J. F.A.K. VAN BENTHEM. AND F. WOLTER (EDS.) , Elsevier , Amsterdam, 2007, pp. 331-426.

8. C. CIRSTEA ET AL., «Modal logics are coalgebraic», *The computer journal*, 54 (2011), 31-41 [attached].

9. P. BLACKBURN, M. DE RIJKE AND Y. VENEMA, *Modal logic. Cambridge tracts in theoretical computer science* , Cambridge UP , Cambridge, UK, 2002 [reprint 2010].

Module 6

The formal ontology of the natural realism (NR) II: Aczel sets and coalgebraic modal logic

Aczel set theory as non-standard set theory

- The main motivation for such a **non-standard set theory** is threefold:

1. As far as standard set theories are aimed at giving a foundation to classical mathematics, mathematical analysis before all, non-standard set theories are aimed at giving a foundation to a **numerical computation** approach to **non-linear/non-integrable systems,** through the simulation of their behaviors.

2. Particularly, Aczel set-theory is aimed at giving a foundation to computation on **infinite data streams** on which no total-ordering can be in principle supposed. [effectively a stream of numbers is a ordered pair in which the first coordinate (*head*) is a number and the second cohordinate (*tail*) a stream of numbers. By a stream is thus possible to represent both equation systems with their results (finite, ordered data streams), but also infinite data streams generated by a finite alphabet (e.g.: by tkaing each time as head a letter, and as tail the sequence of all the possible random combinations of the others)].

3. And at giving a foundation to **program semantics** in TCS, before all **fuctional programming** and the **infinite inclusions** allowed in Kripke relational semantics, so to allow the **behavioral simulation** of **semantic tasks** as far as formalized in modal logic (intensional logics) and hence in **philosophical logic.**

- The main intuition behind Aczel set-theory – originally inspired by De Giorgi's **«Ample Theory» -** is to satisfy all the precedent aims through the removal of the **foundation axiom** from ZFC – present in different fashions in all standard set theories, by a set theory based on an **anti-foundation axiom (AFA)**

Aczel anti-foundation axiom

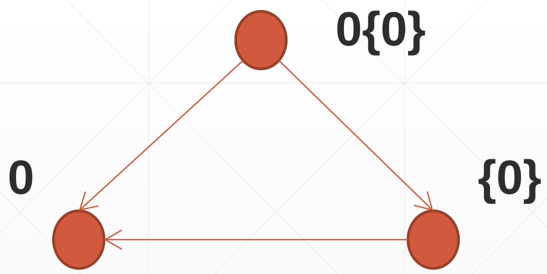
- The foundation axiom in ZF is named also as **axiom of regularity** that states that every non-empty set *A* contains an element that is disjoint from *A*. In first-order logic the axiom reads:

$$\forall x\left(x \neq \varnothing \rightarrow \exists y \in x\left(y \cap x = \varnothing\right)\right)$$

- This means that no well-founded set can include itself and hence that no infinite chain of inclusions is allowed. Through the **axiom of choice** in ZFC – but also with weaker forms of it - this result can be reversed: if there are no such infinite sequences, then the axiom of regularity is true.
- → «The term **non-wellfounded set** refers to sets which contain themselves as members, and more generally which are part of an infinite sequence of sets each term of which is an element of the preceding set. So they exhibit **object circularity** in a blatant way» (Moss 2014).

Aczel sets as directed graphs with a source

- The main result of Aczel's AFA is that each set corresponds to a **directed graph with a source,** where **the edges** represents as many **inclusions relations,** the **nodes** as many (sub-)sets (i.e., each node **is adorned** with a set), and the **source** is the node to which every other node in the graph has a directed path.



Aczel sets and self-inclusion

- Because of AFA allowing **self-inclusions**:

1. The directed graph with only one node and an edge from that node to itself corresponds to a set of the form $x = \{x\}$.
2. A directed cycle graph of length 2 corresponds to a set of the form $x = \{ \{x\} \}$, Etc.

Aczel sets and bisimilarity

- **Bisimulation:** Let $(G,\to)$ be a graph. A relation R on G is a *bisimulation* if the following holds: whenever $x\ R\ y$,

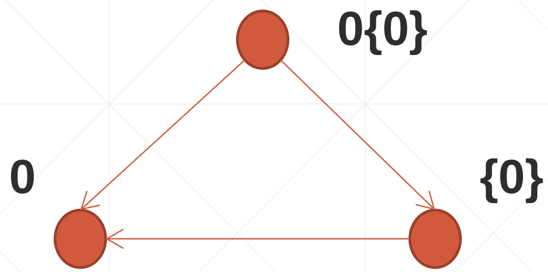
  1. If $x \to x''$, then there is some $y \to y''$ such that $x''\ R\ y''$.
  2. If $y \to y''$, then there is some $x \to x''$ such that $x''\ R\ y''$.

  Of course such a notion can be extended also to to graphs G and H

- When a bisimulation holds for all the relations and nodes between two (or more) graphs the graphs are in a relation of **bisimilarity,** i.e., they are **equivalent for bisimilarity.**

- From the standpoint of algebraic theory this means that the two graphs are in an **homomorphic relation** that is not necessarily an **isomorphism.**

Aczel sets and algebras



- When we see at this graph it is evidently intuitive the **duality** existing between an **algebra** on standard sets in which the transitivity of inclusions is granted by well-ordering and hence where **inductive** procedure are allowed, and a construction on Aczel sets in which the transitivity of the inclusion is not granted even though is not excluded.
- This evicence is at the basis of the interpretation of the **duality in category theory (CT)** between F-algebras/F-coalgebras, where the latter are naturally interpreted as defined on Aczel sets

Category theory and coalgebras I

- CT is a useful theoretical tool, largely used in whichever branch of mathematics and in TCS, to formalize, in a more general and abstract way, very large areas of mathematics (e.g. set theory, group theory, topological spaces, etc.) as **“collections of objects with arrows (morphisms)”.**

- Where a morphism is “a structure preserving mapping between objects of the same category” – in algebra, a “homomorphism”. In this sense, **Set, Grp, Top,** etc. are as many **categories**. From the standpoint of abstract algebra, each category is an algebraic structure including “objects” and “arrows”, so that **Alg** (and **Coalg**) are two other categories, respectively of algebras and coalgebras characterized by “endomorphisms”, from a category into itself. Of course, what characterizes a category is its morphism not its objects, e.g., in **Set** category, objects are sets and morphisms are functions.

- It is important to recall here the warning insert at the beginning of any CT textbook. CT uses many technical terms shared with algebra. Nevertheless, they have in CT **a more abstract and general sense**, because algebras is one of the many categories (**Alg**) with which CT deals with. Therefore, it is important – in the case of ambiguities - to distinguish between the different senses the same term can have, in algebra and in CT.

Category theory and coalgebras II

- More precisely, a category *C* consists of the following three mathematical entities:
- A class **ob**(*C*), whose elements are called “objects”;
- A class **hom**(*C*), whose elements are called “morphisms” or “maps” or “arrows”. Each morphism *f* has a “source object” *a*, and a “target object” *b*.The expression **hom**(*a, b*) – or, equivalently, $\mathbf{hom}_C(a, b)$, **mor**(*a, b*), or **C**(*a, b*) – denotes the **hom**-class of all morphisms from *a* to *b.*
- A binary operation “∘”, called “composition of morphisms”, such that for any three objects *a*, *b*, and *c,* we have **hom**(*b, c*) x **hom**(*a, b*) → **hom**(*a, c*). The composition $f: a \to b$, and $g: b \to c$ is written as $g \circ f$ and is governed by two axioms:
  - *Associativity*: If $f: a \to b$, $g: b \to c$, and $h: a \to b$, then $h \circ (g \circ f) = (h \circ g) \circ f$; and
  - *Identity*: For every object *x,* there exists a morphism $1x: x \to x$ called the “identity morphism for *x*”, id_x, such that for every morphism $f: a \to b$ we have $1_b \circ f = f = f \circ 1_a$.

Category theory and duality

- From the axioms, it is provable that there exists exactly ***one identity morphism*** **for every object,** so that, in many formalizations of CT, **an identity morphism substitutes its own object.**

- There is, of course, a strict relationship between CT and **model theory** in mathematical logic (formal semantics), when we recall that with “category” we intend in formal semantics a set of models, each defined on a given, different domain of objects, but all satisfying (making true) the same syntactic structure (the same formal calculus).

- **Duality** in CT is thus a correspondence between properties of a category C and the so-called “dual properties” of the opposite category C^{op}. This means, roughly speaking, that, given a statement regarding the category C, by interchanging the source and the target of each **functor**, as well as interchanging the order of composing two morphisms, a corresponding dual statement is obtained regarding the opposite category C^{op}. Hence, duality as such is the assertion that **truth/falsehood is invariant under this operation on statements.**

Category theory and duality II

- Formally, let σ be any statement in CT language. We can form the opposite statement σ^{op} by reversing arrows and compositions, i.e., by:

1. Interchanging **source and target** in σ;
2. Interchanging **the order of the two morphism**, i.e., $f \circ g$ with $g \circ f$.

- These two applications of CT duality exemplify, when applied to the duality algebra-coalgebra in the **Alg** category, what TCS scholars mean when they state that “coalgebra is the semantics of the relative algebra”. Applied to the first example, this statement is straightforwardly evident. For the second example, it is sufficient to recall that in set theory and hence in predicate calculus a function (predicate), f: $A \rightarrow B$, is *injective* iff:

$$(\forall a \in A)\big((f(a)=f(b)) \Rightarrow (a=b)\big) \Leftrightarrow \big((a \neq b) \Rightarrow (f(a) \neq f(b))\big)$$

Universal Coalgebra I

- As we know from our previous discussion on duality in CT, what makes so interesting for TCS the algebraic approach to semantics of CS is the possibility of using the first-order language of algebra, for coping with which, in logic, are meta-logical problems!

- This is well-known since early G. Boole's first attempt of developing an algebraic logic. This is much more interesting for us, given that BAO language is at the basis of modern digital computing, and, more generally, of the so-called "equation logic". Indeed, every formula, $p \wedge q$, $p \vee q$, $\neg p$, …, of first-order propositional logic (and calculus), can be translated in a standard way into the corresponding formulas and operations – respectively, $a \wedge b$, $a \vee b$, $-b$, … – of the BAO binary arithmetic. What is important to recall is that they, properly, correspond to as many **semantic evaluations** ("it is true that…") on them – respectively, $\vdash p \wedge q$, $\vdash p \vee q$, $\vdash \neg p$, $\vdash$…, or, in the symbolism of universal algebra:

$$(\llbracket \varphi \rrbracket, \llbracket \psi \rrbracket, \llbracket \varsigma \rrbracket, \llbracket \ldots \rrbracket)$$

Universal Coalgebra II

- for each of the first-order propositional formulas considered. In other terms, Boolean operators (Λ,V,–) correspond to operations *on sets* , *not on individuals* like the related propositional connectives (∧,∨¬).
- Of course, for a proper bidirectional translation between propositional formulas and algebraic equations, we have to enrich the usual set of Boolean operators,

  $(\cup\cap^{-})$

  also with the two operators of "truth" (⊤) and falsehood (⊥), so to complete the list of BAO operators according to the following: (⊤,⊥,Λ,V,–).
- The problem is that BAO, works on *finite sets*, because of the "constructive" character of its operations. For developing a proper semantics of the propositional calculus using the first-order formal language of algebra and CT, we need to extend the semantic use of BAO to infinite sets semantic evaluation. This can be done by using the powerful tool of CT duality, between *C* and C^{op}, in the case, between **Alg** and **Coalg**. As we know, duality grants the truth/falsehood (⊤/⊥) invariance, for the interchange of the source and target of each algebraic morphism, as well for the interchange of the order of their composition. That is: **Alg** $\equiv_{\top/\bot}$ **Coalg.**

Universal Coalgebra III

- In this way, it is possible to develop, on the algebraic side, a BAO logic also of modal logic. I.e., an algebraic theory of modal logic that is *dual* w.r.t. ordinary modal propositional logic, but much more powerful. Before all because it is possible to demonstrate, in an algebraic theory of modal logic, besides many other important results, the ***completeness* of modal semantics**, which it is impossible to obtain in classical relational semantics (Kripke's modal semantics for infinite model inclusions).

- On the contrary, on the coalgebraic side of the duality, as far as defined on **Aczel sets** allowing **infinite inclusions**, the coalgebraic perspective on modal logic implies a ***generalization* of modal results**, starting from the possibility of formalizing in it not only the notion of **equivalence by *bisimulation*** – already applied to Kripke's frames in relational semantics.

Universal Coalgebra IV

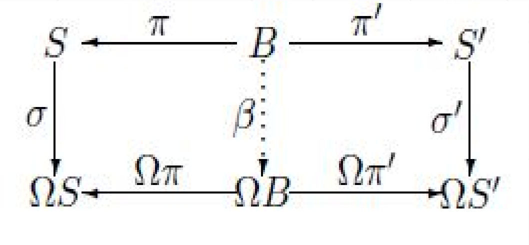
- Overall, it becomes possible to extend the notion of bisimulation equivalence to the ***behavioural* or *observational equivalence* notion**. This, allows to construct the theory of Universal Coalgebra as dual to Universal Algebra, and hence as a *general theory of dynamic systems* [15], quantum STS's included. Indeed, in quantum physics, like in the infinite-state non-deterministic automata, we can deal only with the *observables* of the dynamics, without any possibility of "seeing inside" the black-box of a quantum system.

- To understand intuitively, it is sufficient, in TCS interpreting **the edges of Aczel sets** in terms of **labelled state transitions** (where the label is a given function of the program, from which the notion of automata as STS), and in quantum physics as the **operators defined on an Hilbert space –** the main "quantum observables" – for governing the transitions among **quantum states.**

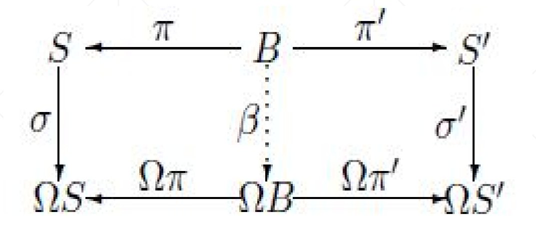
Universal Coalgebra V

- In such a way, we are able to define now the notions of bisimulation and behavioural (observational) equivalence in coalgebraic terms. Let us start with the notion of bisimulation:

Def.2: Let S = $<S, \sigma>$ and **S**' = $<S', \sigma'>$ be two systems for the set functor Ω. A relation $B \subseteq S \times S'$ is a *bisimulation between* **S** and **S'**, if we can endow it with a coalgebra map $\beta : B \rightarrow \Omega B$, in such a way that the two projections $\pi : B \rightarrow S$ and $\pi' : B \rightarrow S'$ are homomorphisms from $<B, \beta>$ to S and S**'**, respectively:



Universal Coalgebra VI



- If there exists a bisimulation *B* with (*s,s'*) $\in$ *B,* we say that *s* and *s'* are *bisimilar,* in symbols: S,*s* $\underline{\leftrightarrow}$ S',*s'* (or *B*: S,*s* $\underline{\leftrightarrow}$ S',*s'*, for making *B* explicit).
- Finally, if S = S' we say that *B* is a bisimulation on S. If this *B* happens to be an *equivalence relation,* so that we call it a *bisimulation equivalence* on S.
- A much more powerful notion for STS and dynamic system coalgebraic interpretation is the notion of *behavioural* or *observational equivalence,* formalizing the intuitive idea that two automata (systems) are equivalent if their behaviours are indistinguishable, that is, if we cannot distinguish them by observation.
- **Def. 3:** In the case that the functor Ω admits a final coalgebra Z such that from every coalgebra A in **CoalgΩ** there is a unique homomorphism $!_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{Z}$

Coalgebraic modal semantics and biconditionals

- Finally, the most important point is the possibility of justifying in a coalgebraic modal semantics a **logical (truth-functional) arrow reversal** for which the bisimilarity equivalence based on homomorphism is not suffcient.
- This can be obtained, whereas the use of the (strong) bi-conditional is not allowed through the notion of **p-morphism** or **bounded morphism:**

DEFINITION 15. Let $\mathfrak{M} = \langle W, \{R_\alpha\}_{\alpha\in\tau}, V\rangle$ and $\mathfrak{M}' = \langle W', \{R'_\alpha\}_{\alpha\in\tau}, V'\rangle$ be Kripke structures. A function $\rho\colon W \to W'$ is a *bounded morphism* from $\mathfrak{M}$ to $\mathfrak{M}'$ if its graph is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. We denote a bounded morphism as in $\rho\colon \mathfrak{M} \xrightarrow{\rightleftarrows} \mathfrak{M}'$.