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A Dual Ontology of Nature, Life, and Person

Unit 10: The doubling of the Hilbert space and the categorical duality between q-deformed Hopf coalgebras/algebras

Course WI-FI-BASTIONTO-ER

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By

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The doubling of the Hilbert space and the categorical duality between *q*-deformed Hopf coalgebras/algebras

The (Co)Algebraic Interpretation of QFT as q-deformed Hopf Algebra / Coalgebra

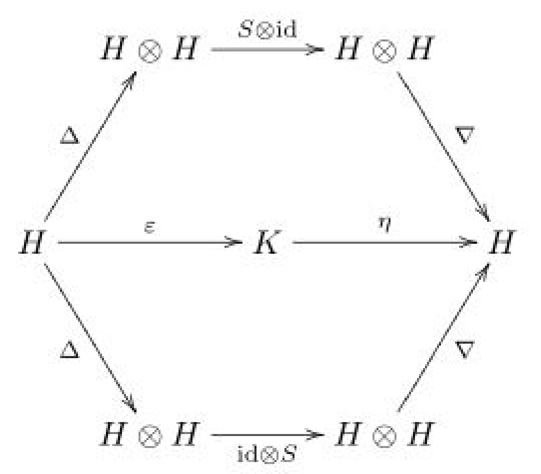
Hilbert space and Dirac Delta Function in the QM interpretation of quantum systems

- As we know, for the Heisenberg uncertainty principle, the canonical variables of Newtonian mechanics **position, *x*,** and **momentum, *p*, do not commute between themselves** for quantum particles in QM → necessarily statistical and not geometrical representation of quantum states → **Schroedinger wave function**.
- → However the outstanding Hilbert's discovery that the canonical variables **commute also in QM each with the Fourier transform of the other →** notion of **canonical commutation relations** (CCR).
- This opened the way to a geometrical representation also of a QM system in terms of a **Hilbert space** to which it is possible to apply the powerful algebraic tool of the **vectorial calculus,** and hence of the **matrix calculus** of the classical statistical mechanics to the wave function of QM.

Continuing…

- However, quantum calculations are so faced with the problem of **undesired infinite magnitudes** in two ways:
  1. On one side, Hilbert vectorial space is necessarily infinite-dimensional → Stone-Von Neumann theorem (1931) **a finite number of unitarily equivalent CCR's** is necessary and sufficient for representing a QM system → choice of a **finite orthonormal basis of the Hilbert space** sufficient for representing adequately a given quantum system **depends on the human observer** → epistemological problem of the intrinsic **observer-dependent character** of QM calculations;
  2. On the other side, in QM **matrix calculus** the infinity appears with the problem of the so-called **Dirac Delta Function:** if I want to reduce the statistical variance on one of the two correlated magnitudes, I have to suppose necessarily **infinite** the other one, and vice versa → usage of the so-called **renormalization groups** for solving the problem ('tHooft).
- Generally the algebraic tool for performing calculations on a lattice of quantum numbers is the **Hopf Algebra** that is a **bialgebra** composed by an **algebra** *A* x *A* → *A* and a **coalgebra** *A* → *A* x *A* perfectly **isomorphic,** linked by a **linear mapping** on a vector space. Because both **products** (algebra) and **coproducts** (coalgebra) pairs **commute within themselves** they are **covariant → Hopf algebra is self-dual**

Hopf bialgebra and its self-duality



The dynamic choice by the system of its representation space in QFT

- On the contrary, **in thermal QFT**, given that, each quantum system corresponds to one of the indefinitely many "spontaneous symmetry breakdowns" (SSB) of the QV, **splitting locally the QV into a thermodynamic pair system-thermal bath**, the proper mathematical formalism for quantum calculations in QFT are based on ***q –deformed* Hopf co-algebras that are contravariant and hence dual as to the corresponding *q –deformed* Hopf algebras** because the *q-deformation* parameter that is a **thermal parameter** broke the symmetry of a Hopf bialgebra**.**

- Coproducts of ***q –deformed* Hopf co-algebras** indeed **do not commute** between themselves, since one term represents a system state, the other one a "mirroring" thermal bath state, and then cannot be represented on the same basis.

- *q* indeed is a **thermal parameter**, linked to **the Bogoliubov transform**, i.e., linked to the so-called **"operator of particle annihilation-creation in the QV"** (G. Vitiello *et al.*).

Bogoliubov transform and the dissipative QFT

- We offer in Basti2 a **coalgebraic interpretation of modal logic,** where, as far as defined on Aczel's **non-well founded sets,** it is able to justify the existence of **homomorphisms between coalgebras** in terms of **bisimilarity and/or of observational equivalence,** and a **coalgebraic theory of proof** based on the powerful notion of **coinduction** as dual to algebraic induction.
- As to the present aim of justifying a **dissipative QFT** as **the proper QFT** because satisfying Haag's theorem, we have to introduce the key notion of Bogoliubov transform (BT), that offers an essential contribution to the study of **the commutation/anticommutation relations for bosonic/fermionic creation and annnihilation operators** in a QV with a **temperature >0.**
- Such a transform, since it was introduced by the Russian mathematician **Nikolay Bogoliubov** (1929-1992) in 60's**, is a fundamental ingredient of QFT** in condensed matter physics, with applications in **superconductivity** and **ferromagnetism,** that is for studying **system able to pass through several stable phases,** as far as these passages are strictly related with the **thermal parameter, θ.**
- More recently such a transform had an essential role in demonstrating the **existence of Hawking radiation for the black-holes,** and hence with the emergence of the **Berkenstein-Hawking entropy** we already introduced, as an essential ingredient of present cosmology**.** What is more important for us is that, after the action of Bogoliubov operators of creation/annihilation, **the QV ground state |0⟩** is modified by the determination in it of **squeezed coherent states** correspondent to as many **phase coherence phenomena in it.**

q-deformed Hopf algebras in a thermal QV condition

- As Vitiello emphasizes, all this explains why the squeezed coherent states intrinsically related to the **q-deformed Hopf algebras** were irrelevant, when introduced in the context of the **closed quantum systems** of QM, or of the algebraic interpretation of QFT studied in the precedent module.
- Without a thermal vacuum, the Stone-Von Neumann theorem holds, and the infinitely many inequivalent representations of the CCR cannot occur, despite they are previewd by the Haag theorem, are related with the **spontaneous symmetry breakdowns (SSB)** of the QV, and were firstly revealed and defined rigorously by the BT creation/annihilation operators,.
- All this means that the **q-parameter** of the Hopf algebra deformation must be identified with the **thermal parameter θ of the BT,** so to allow the interpretation of the **duality q-deformed Hopf algebras/coalgebras,** with the intrinsic **reversal of the arrows and of the composition** just introduced, in terms of a **thermodynamic energy balance,** giving coalgebras formalism an essential role, not only in TDS, but in **fundamental physics.**

The Doubling of the Degrees of Freedom (DDF) and the QV-foliation in QFT of dissipative systems

- In the corresponding Hilbert space is then **doubled** because the non-commutativity of coproducts implies that at each state of the system corresponds the **mirroring state of the thermal bath** i.e., the Hamiltonian character of the system is recovered by inserting systematically the thermal bath in the Hilbert space.
- I.e., limiting ourselves to the bosonic case, so that working on the hyperbolic function basis $\{e^{+\theta}, e^{-\theta}\}$, we obtain the commuting operators acting on this doubled Hilbert space given by the application of the Bogoliubov transform
- They give a concrete realization of the vectorial mapping of the *q-deformed* Hopf coalgebra: $A \to A \times A$

$$A(\theta) = A\cosh\theta - \tilde{A}^{\dagger}\sinh\theta, \tag{4}$$

$$\tilde{A}(\theta) = \tilde{A}\cosh\theta - A^{\dagger}\sinh\theta. \tag{5}$$

The canonical commutation relations are

$$[A(\theta), A(\theta)^{\dagger}] = 1, \qquad [\tilde{A}(\theta), \tilde{A}(\theta)^{\dagger}] = 1, \tag{6}$$

- Because each of the system represents a QV local degeneracy at the ground state, it is very robust → principle of the **QV-foliation each labelled by a *q-value*** with the corresponding foliation of the doubled Hilbert space = **robust dynamic mechanism of memory and construction used by nature**

Minimum of free energy as an evaluation function (Boolean operator) of the doubled qubit

- In fact, on the basis of the QFT duality principle, it is the dynamic system (coalgebra) that chooses how many terms there are, and then **maps this choice** on the algebra → **it is the dynamic (not the observer) that chooses the orthonormal basis of the Hilbert space** composed by "doubled terms" → i.e., the principle of the **doubling of the degrees of freedom between algebra and coalgebra.**

- The contravariance between algebra and coalgebra **with the reversal of all the arrows and the compositions has therefore a (thermo-)dynamic control: the energy balance = minimum of the free energy when the two subsystem are prefectly matching between each other.**

- On this basis it is possible to design a revolutionary architecture of quantum computer based on QFT where the maximum of entropy (minimization of free-energy) plays the role of **a first-order evaluation function** for the **local** semantics, implemented in the dual coalgebra of the corresponding Boolean Algebra (i.e., notion of a semantic q-bit in QFT computing *vs.* the syntactic q-bit of QM computing).