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A Dual Ontology of Nature, Life, and Person

Unit 8: Some elements of Category Theory (CT), and the notion of categorical dual equivalence in mathematics and logic

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By

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Bibliography

Some references

- All the information and the necessary references can be found in:
- **Main Reference:**
  - Basti, G.. The Post-Modern Transcendental of Language in Science and Philosophy. In Z. Delic (Ed.), *Epistemology and Transformation of Knowledge in Global Age* (pp. 35-62). London: InTech, 2017. doi:10.5772/intechopen.68613 [attached]
- **Other Reference:**
  - Abramsky, S., & Tzevelekos. Introduction to categories and categorical logic. In B. Coecke (Ed.), *New structures for physics. Lecture Notes in Physics, vol. 813* (pp. 3-94). Berlin-New York: Springer, 2011 [attached]

The logical and mathematical foundation of QFT calculation in Category Theory (CT)

- What is CT? It is a natural recovering of Ch. S. Peirce and E. Schroeder research project of an **algebraic foundation of logic and mathematics** (algebra of relations, calculus of relations) suddenly interrupted by the **Loewenheim-Skolem Theorem** (1921) → migration of set-theoretic semantics (Boolean Algebra semantics included) **to second (and higher) order logic** → introduction of arbitrariness («creativity», for being more politically correct) in mathematics, because second (and higher) order predicate calulus is not **complete** differently from first-order one (Cfr. Goedel demonstration (1929), as necessary premise of his famous **incompleteness theorems for first order theories (models) in 1931**). These are the «strong roots of the weak thought», as I entitled one book of mine on these topics.

- **Anti-platonic (**and anti-modern) **stance (**implicitely **Aristotelian)** of CT: even set elements are domain-codomain of arrows (morphisms) (Abramsky) → natural place for developing a set-theoretic modal logic in its **pragmatic** interpretation by Peirce and, obviously, for giving a full foundation of **operator logic** also in quantum systems.

Some elements of CT

- the *primitives* are: 1) **morphisms or arrows**, *f, g,* – intended as a generalization of notions such as "function", "operator", "map", etc., –, 2) the **compositions of arrows,** $f \circ g$; and 3) **two "mappings", *dom*(•), *codom*(•)** assigning a domain and a codomain to each arrow.

- Any object *A, B, C,* characterizing a category, can be substituted by the correspondent **reflexive morphism** $A \rightarrow A$ constituting **an identity relation** Id_A. Moreover, for each triple of objects, *A,B,C,* there exists a *composition map* $A \xrightarrow{f} B \xrightarrow{g} C$ written as $g \circ f$.

- **Therefore, a *category* is any structure in logic or mathematics with structure-preserving morphisms**. E.g., in set theoretic semantics, all the models of a given formal system, because sharing the same structure, constitute a category.

- In this way, some fundamental mathematical and logical structures are as many **categories**: **Set** (sets and functions), **Grp** (groups and homomorphisms), **Top** (topological spaces and continuous functions), **Pos** (partially ordered sets and monotone functions), **Vect** (vector spaces defined on numerical fields and linear functions), etc.

Partial and Total Orderings as Categories

- Particularly, the category of **Pos** is fundamental in logic. Indeed, partial ordering is a structure of ordering relations, $\leq$, among sets satisfying simultaneously:
  - $x \leq x$ (Reflexivity)
  - $x \leq y \wedge y \leq x \Rightarrow x = y$ (Antisymmetry)
  - $x \leq y \wedge y \leq z \Rightarrow x \leq z$ (Transitivity)
- The structure of "total ordering" of sets, and the relative category **Tos** of totally ordered sets, satisfies Antisymmetry and Transitivity but instead of Reflexivity it satisfies the ordering property:
  - $x \leq y \vee y \leq x$ (Totality)
- That is, for all sets an ordering relations is defined

Partial and Total Orderings in Set Theory and CT

- In set theory, the total orderings include the partial ones, since totality includes reflexivity.
- This is not the case of CT where, **because of its anti-Platonic stance**, object existence is not supposed to relations, so that for existing an object has to satisfy a reflexivity relation. In other terms, in CT total relations have to be attentively justified never supposed.
- Therefore, the category **Tos** lacks in an "object" as to **Pos,** because the ordering relation $\leq$ that **Tos** uses is no longer an object in it, since does not satisfy any longer reflexivity like in **Pos**. Therefore, **Tos** is a subcategory of **Pos.** In fact, fundamental posets are the real number set $(\mathbb{R},\leq)$, and the power set $\mathcal{P}$ of a given set X $(\mathcal{P}(X),\subseteq)$.

Continuing…

- Another fundamental notion in CT is the notion of **functor,** *F,* that is, an operation mapping objects and arrows of a category **C** into another **D**, *F:* **C** → **D,** so to preserve compositions and identities. In this way, between the two categories there exists a ***homomorphism up to isomorphism***. Generally, a functor *F* is **covariant***,* that is, it preserves arrow directions and composition orders (e.g., in the QM attempt of interpreting thermodynamics within kinematics (Connes e Rovelli)), i.e.:

$$\text{if } f : A \to B, \text{ then } FA \to FB; \text{ if } f \circ g, \text{ then } F(f \circ g) = Ff \circ Fg; \text{ if } id_A, \text{ then } Fid_A = id_{FA}.$$

- However, two categories can be equally homomorphic up to isomorphism if the functor *G* connecting them is **contravariant,** i.e., **reversing** all the arrows directions and the composition orders, i.e. *G*: **C** → **D**$^{\mathbf{op}}$:

$$\text{if } f : A \to B, \text{ then } GB \to GA; \text{ if } f \circ g, \text{ then } G(g \circ f) = Gg \circ Gf; \text{ but if } id_A, \text{ then } Gid_A = id_{GA}.$$

Continuing…

- Through the notion of contravariant functor, we can introduce the notion of *category duality.* Namely, given a category **C** and an *endofunctor E*: **C** $\rightarrow$ **C** on a category onto itself, the contravariant application of *E* links a category to its opposite, i.e.: E^{op}: **C** $\rightarrow$ $\mathbf{C}^{\mathbf{op}}$.

- In this way it is possible to demonstrate the *dual equivalence* between them, in symbols: $\mathbf{C} \rightleftharpoons \mathbf{C}^{\mathbf{op}}$. In CT semantics, this means that given a statement α defined on **C** α is true *iff* the statement α^{op} defined on $\mathbf{C}^{\mathbf{op}}$ is also true.

- In other terms, truth is invariant for such an exchange operation over the statements, that is, they are *dually equivalent.* In symbols: $\boldsymbol{\alpha} \rightleftarrows \boldsymbol{\alpha}^{\mathbf{op}}$, as distinguished from the ordinary equivalence of the logical tautology: $\alpha \leftrightarrow \beta$, defined within the very same category, i.e., on the same basis.

Duality and semantics

What is important to emphasize for our aims is that in computational set theoretic semantics the dual category of $\mathbf{Set^{op}}$ is more important than $\mathbf{Set}$, given that, for instance a generic conditional in logic "if...then", e.g., "for all x, if x is a horse, then x is a mammalian", is true iff the "mammalian set" includes the "horse set" with all its subsets. Therefore, the semantics of a given statement is set theoretically defined on the powerset $\mathcal{P}(X)$ of a given set X. From this the categorical definition of the *covariant* powerset functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ derives:

Definition (Categorical definition of the covariant powerset functor). The powerset functor $\mathcal{P}$ is a covariant endofunctor $\mathbf{Set} \to \mathbf{Set}$, mapping each set X in its powerset $\mathcal{P}(X)$ and sending each function $f : X \to Y$ to the map S that sends $U \subseteq X$ to its image $f(U) \subseteq Y$, that is:

$$\begin{gathered} X \mapsto \mathcal{P}(X). \\ (f : X \to Y) \mapsto \mathcal{P}(f) := S \mapsto \{f(x)|x \in S\} \end{gathered} \tag{B1}$$

Conversely, the definition of the contravariant power set functor $\mathcal{P}^{op} : \mathbf{Set^{op}} \to \mathbf{Set}$ is as following:

Definition (Categorical definition of the contravariant power set functor). The contravariant power set functor $\mathcal{P}^{op}$ is a contravariant endofunctor $\mathbf{Set^{op}} \to \mathbf{Set}$ that preserves all the objects, and sends each function $f : X \to Y$ to the map T sending $V \subseteq Y$ to its inverse image $f^{-1}(V) \subseteq X$. Therefore:

$$\begin{gathered} \mathcal{P}^{op}(X) := \mathcal{P}(X). \\ \mathcal{P}^{op}(f : X \to Y) : \ \mathcal{P}(Y) \longrightarrow \mathcal{P}(X) := T \mapsto \{x \in X | f(x) \in T\} \end{gathered} \tag{B2}$$

Another useful example for us, because used, both in the GNS-construction for C^*-algebras , and in the Vietoris construction for Boolean Algebras, is the dual space functor $(\_)^*$ on vector spaces V defined on a field k:

$$(\_)^* : \mathbf{Vect}_k^{op} \to \mathbf{Vect}_k := V \mapsto \ V^* \tag{B3}$$

Other Useful Dual Constructions in CT

- Moreover, other useful categorical dual constructions can be significantly formalized in CT that we cannot define here, but that have an immediate significance for us because both the topological formalism of quantum physics and of quantum computation are plenty of exemplifications of their usage. For instance, the notions of **“left” and “right adjoints”** of functions and operators, the notions of **“universality” (uniqueness) and “couniversality”,** of **“products” and “coproducts”,** of **“limits” and “colimits”** interpreted, respectively, as **“final” and “initial” objects of two categories** related by a third category of **“indexing functors”,** so to grant the mapping, via **a “diagonal functor**”, of all the objects and morphisms of one category into the other.

- Practically, all the objects and the operations that are usefully formalized in set theory, and then in calculus and logic – included the **“exponentiation” operation** for forming function spaces, and the consequent “evaluation function” over function domains –, can be usefully formalized also in CT, **with a significant difference**, however. Instead of considering objects and operations for what they “are” as it is in set theory, in **CT we are considering them for what they “do”** [37, p. 53], so to fulfil in formal way the **primacy of *pragmatics* over syntax and semantics** that the semiotic interpretation of logic by Moore borrowed from his teacher Peirce.