Uniwersytet
Kardynała Stefana Wyszyńskiego
w Warszawie

A Dual Ontology of Nature, Life, and Person

Unit 9: The Stone theorem and the coalgebraic semantics of Boolean algebras: its relevance in logic and computer science

Course WI-FI-BASTIONTO-ER 2017/18

By

GIANFRANCO BASTI

Full Professor of Philosophy of Nature and of Sciences
At the Faculty of Philosophy of the Pontifical Lateran University

E-mail: basti@pul.it
Address: Pontifical Lateran University – Piazza S. Giovanni Laterano, 4 – 00184 Rome
Phone: +39 06 69895656
Cell.: +39 339 5760314
Web: www.irafs.org

Bibliography

Some references

- **Main references:**
  - Basti, G.. The Post-Modern Transcendental of Language in Science and Philosophy. In Z. Delic (Ed.), *Epistemology and Transformation of Knowledge in Global Age* (pp. 35-62). London: InTech, 2017. doi:10.5772/intechopen.68613 [attached].
  - Basti, G.. The quantum field theory (QFT) dual paradigm in fundamental physics and the semantic information content and measure in cognitive sciences. In G. Dodig-Crnkovic, & R. Giovagnoli (Eds.), *Representation and Reality in Humans, Other Living Organisms, and Intelligent Machine* (pp. 177-210). Berlin, New York: Springer Verlag. doi:10.1007/978-3-319-43784 [attached].
  - [See also the wide bibliography quoted in these two papers]

Coalgebraic Logic in the "Stone Era" of QC

- Effectively, one of the pillars of the topological quantum computing, i.e., the fundamental **Stone representation theorem for Boolean algebras** (Stone, 1936), is strictly related with the topological interpretation of QFT.
- Stone arrived at this theorem after having demonstrated in 1931, with J. von Neumann, a fundamental theorem at the basis of the formalism for QM (von Neumann, 1955).
- Stone theorem states that a Boolean Algbera (BA) is **isomorphic with a ultrafilter of clopen sets** defined on a **Stone topological space**.
- The connection between 1931/1936 theorems is that the topologies of the Stone spaces **are the same** of the topological spaces associated with C*-algebras in QFT via the GNS-construction, even though this construction is dated several years (1943-1947) after Stone theorem (Landsmann 2011;Basti et al. 2017).

Stone Spaces and Colagebraic Logic

- Moreover, the category of Boolean algebras **BAlg** and the category of the Stone-spaces **Stone** are **dual** as to each other, in the sense that, given two Boolean algebras *A, B*, and the respective Stone-spaces *S*(*A*), *S*(*B*), to a continuous function from *S*(*A*) to *S*(*B*) it corresponds a monotone function from *B* to *A*.

- Afterwards, a seminal work of S. Abramsky at the end of the 80's (Abramsky, 2005), demonstrated in CT logic the **dual equivalence** between the category of coalgebras defined on Stone-spaces **SCoalg**, and the category of Boolean algebras **BAlg**, , i.e..

$$\mathbf{SCoalg}(\mathcal{V}) \rightleftharpoons \mathbf{BAlg}(\mathcal{V}^*)$$

for the contravariant application of the so-called **Vietoris functor** $\mathcal{V}$**.** I.e., by a contravariant vectorial mapping using the so-called “Vietoris construction”, where the Vietoris space is a vector space sharing the same topology of a Stone space.

- In this way, we introduce the possibility of defining on the physical states of a (quantum) system coalgebraically modeled, the semantics of computing systems, with evident advantages for **functional programming** (Rutten, 2000; Venema, 2007; Abramsky 2013).

Stone Coalgebras over NWF Sets

- The momentous result of Abramsky depends on defining coalgebras over P. Aczel's **non-well founded (NWF) sets** (Aczel, 1988), that is, a set theory where the regularity axiom of the ZF (Zermelo-Fraenkel) set theory does not hold, and then **no set total ordering is allowed**.

- In fact, because of P. Aczel's **anti-foundation axiom (AFA)**, the set self-inclusion is possible, and then **unbounded chains of set inclusion** are allowed.

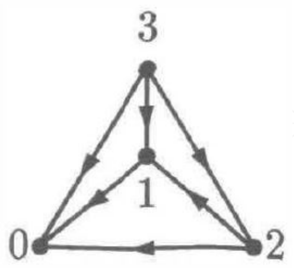
- NWF sets are then what is necessary for a **coalgebraic, set-theoretic modeling of infinite streams**. Indeed, in NWF-set theory the powerful **final coalgebra theorem** holds (Aczel and Mendel, 1989).

- It affirms that in NWF-set theory, all the sets represented as oriented pointed graphs (effectively, accessible pointed graphs, APG) are **trees with the same ultimate root.** In this sense the **ultimate root** is similar to the universal **collection *V*** of standard set theory, with the mathematical ontology difference that in the ultimate root all the set and their elements does not exist **actually** given that there is no total ordering, but **virtually,** as far as they are **unfolded** from their common root.

- In fact, the lacking of total ordering allows different and irreducible paths of **subsets unfolding** so that, properly, in such a coalgebraic structure, a set does not include univocally – i.e. according to the linear transitive inclusion rule – its subsets, but only a set **admits (not includes) its subsets,** following an Euclidean transitive rule where no jump ascendant-descendant is allowed, so to define a tree-like process of unfolding of several reciprocally non-ordered subsets from the same superset.

AFA explanation

- **AFA: every graph (APG: accessible pointed graph) has a unique decoration**
- For instance in the case of the apg of the number 3, the decoration is the following:



- The node labelled 0 has no children and hence must be assigned the empty set, i.e. 0, in any decoration. The central node has as only child the node labelled 0. Hence in any decoration the central node must be assigned the set {0}, i.e. 1.
- **→ *Every well-founded graph has a unique decoration* → every well-founded apg is the picture of a unique set.**
- But there exist pictures also of **non-wellfounded sets.** To see this we will associate with each set *a* its **canonical picture**. Form the graph that has as its nodes those sets that occur in sequences a^0, a^1, a^2, ... such that: $. . . \in a^2 \in a^1 \in a^0 = a$, and having as edges those pairs of nodes (x, y) such that $y \in x$.

AFA explanation

- **Such an apg does not require a well-founded set.** Indeed, every picture of a set **can be unfolded into a tree picture of the same set**. Given an apg we may form the tree whose nodes are the finite paths of the apg that start from the point of the apg and whose edges are pairs of paths of the form

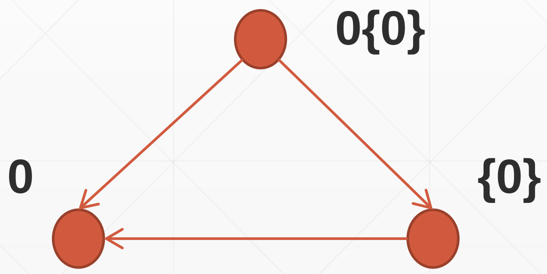
$$a^0 \rightarrow \ldots \rightarrow a,\ a^0 \rightarrow \ldots \rightarrow a \rightarrow a'$$

- The root of this tree is the path a^0 of length one. This tree is the **unfolding** of the apg. Thus the unfolding of an apg will picture any set pictured by the apg. The unfolding of the canonical picture of a set will be called the **canonical tree picture** of the set. From this construction the AFA derives immediately:

"Every graph has a unique decoration"

Aczel sets as directed graphs with a source

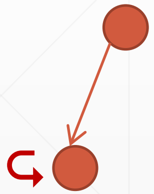
- The main result of Aczel's AFA is that each set corresponds to a **directed graph with a source (root),** where **the edges** represents as many **inclusions relations,** the **nodes** as many (sub-)sets (i.e., each node **is adorned** with a set), and the **source** is the node to which every other node in the graph has a directed path.
- Below there is the inclusion path of the number two, following the Euclidean transition rule (*xRy, xRz* → *yRZ*).



Aczel sets and self-inclusion

- Because of AFA allowing **self-inclusions**:

1. The directed graph with only one node and an edge from that node to itself corresponds to a set of the form $x = \{x\}$.
2. A directed cycle graph of length 2 corresponds to a set of the form $x = \{\{x\}\}$.
3. A directed graph of length 2 corresponding to a set of the form $z = \{y, \{x\}\}$



Aczel sets and bisimilarity

- **Bisimulation:** Let $(G,\to)$ be a graph. A relation R on G is a *bisimulation* if the following holds: whenever $x\ R\ y$,

  1. If $x \to x''$, then there is some $y \to y''$ such that $x''\ R\ y''$.
  2. If $y \to y''$, then there is some $x \to x''$ such that $x''\ R\ y''$.

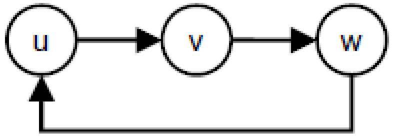
  Of course such a notion can be extended also to to graphs G and H

- When a bisimulation holds for all the relations and nodes between two (or more) graphs the graphs are in a relation of **bisimilarity,** i.e., they are **equivalent for bisimilarity.**

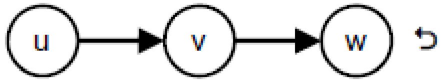
- From the standpoint of algebraic theory this means that the two graphs are in an **homomorphic relation** that is not necessarily an **isomorphism.**

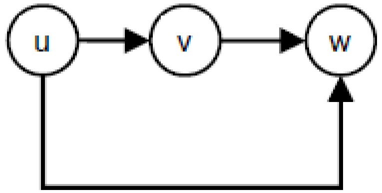
Continuing…

- Indeed, because of the powerful **Final Coalgebra Theorem** (Aczel e Mendler), it is possible to demonstrate the existence of an ultimate **"root"** of all the NWF sets, → Possibility of defining **freely** "trees" of set inclusions, <u>because no-total ordering in NWF sets</u> → causal sets by coinduction

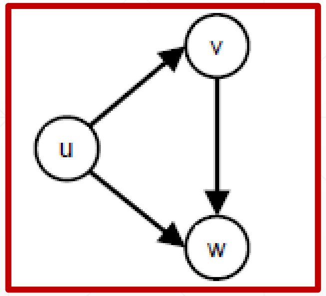


Serial Rule

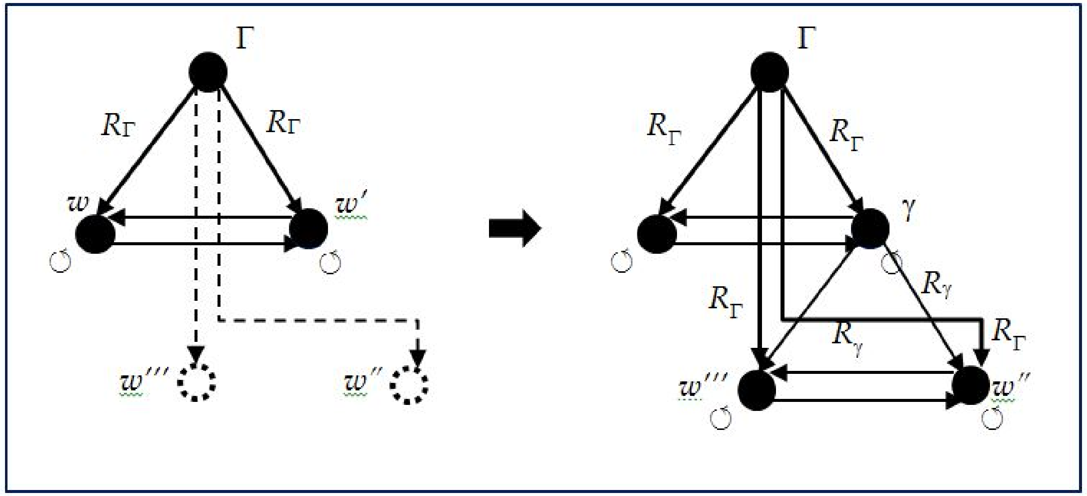




Transitive rule in standard totally ordered sets TOS (E.g., ZF set theory)



Transitive Euclidean Rule in NWF sets where only POS are allowed: **what we need for BAlg!!!!**



Co-induction, in NWF set coalgebras, of **equivalence classes by causal trees** where each descendant has its own ascendant, and where **different choices** are allowed because no TOS!

Categorical Duality SCoalg and BAlg and the notion of Universal Coalgebra

1. **Dual equivalence of the respective categories, by a contravariant vectorial mapping $\mathcal{V}$*** ("Vietoris construction") from the category of coalgebras of NWF-sets defined on Stone spaces, **Scoalg,** onto the category of Boolean Algebras, **BAlg.** That is, **SCoalg**$\mathcal{V} \leftrightarrows$ **BAlg**$\mathcal{V}$***** → ⊤/⊥ operators of BA algebraically **lower bounded** and coalgebraically **upper bounded** → notion of **finitary computation → coalgebraic semantics of Boolean logics** (and then of propositional and predicate calculus) (Abramsky 1988; Venema 2007). Notion of **bounded morphism** for the dual equivalence in **Kripke model** semantics:

$$\Box_{n|\forall n(n>m)}\left(\underbrace{horse \in mammmalian}_{\text{Algebra}(\Omega^*)} \quad \underbrace{\xleftarrow{\rightleftarrows}}_{\text{Bounded Morphism}} \quad \underbrace{horse \ni mammalian}_{\text{Co-Algebra}(\Omega)}\right)$$

2. **Definition of "Universal Coalgebra" as dual to "Universal Algebra"** and interpreted as **general theory of systems** interpreted as **labelled state transition systems (LTS)** (Rutten 2000) → construction of the **"infinite state black-box machine"** for modelling infinite streams through the **dual equivalence** between a final coalgebra and an initial algebra.

- → Applicability of this computational logic to a new class of quantum computer based on **thermal QFT** (Basti, Capolupo & Vitiello 2017) where the minimum of free energy grants the functorial homomorphism coalgebra/algebra.